PRINCIPLES OF ANALYSIS
TOPIC 2: PARTIALLY ORDERED SETS

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Definition 1. A *relation* on a set A is a subset $\bowtie \subset A \times A$. We write $a_1 \bowtie a_2$ to mean $(a_1, a_2) \in \bowtie$.

Let A be a set, $\bowtie$ a relation on A, and $B \subseteq A$. The relation $\bowtie$ restricted to B is $\bowtie_B = \bowtie \cap (B \times B)$; then $\bowtie_B$ is a relation on B which we may also denote by $\bowtie$.

Definition 2. A *partial order* on a set A is a relation $\preccurlyeq$ satisfying

(PO1) $a \preccurlyeq a$ (reflexivity)
(PO2) $a \preccurlyeq b$ and $b \preccurlyeq a$ implies $a = b$ (antisymmetry)
(PO3) $a \preccurlyeq b$ and $b \preccurlyeq c$ implies $a \preccurlyeq c$ (transitivity)

A *partially ordered set* $(A, \preccurlyeq)$ is a set A together with a partial order $\preccurlyeq$ on A.

Example 1. Let $\mathbb{N} = \{1, 2, 3, \dots\}$ denote the set of natural numbers, and let $\leq$ denote the standard order relation on $\mathbb{N}$. Then $(\mathbb{N}, \leq)$ is a partially ordered set.

Example 2. Let X be a set and let $\mathcal{P}(X)$ denote the *power set* of X, that is, $\mathcal{P}(X)$ is the set of all subsets of X. Let $\subseteq$ denote setwise containment. Then $(\mathcal{P}(X), \subseteq)$ is a partially ordered set.

Example 3. Let $n \in \mathbb{N}$ and let $\mathrm{Div}(n)$ denote the set of all positive integers which divide n. For $a, b \in \mathbb{N}$, say that *a divides b*, and write $a \mid b$, if $b = am$ for some $m \in \mathbb{N}$. Then $(\mathrm{Div}(n), |)$ is a partially ordered set.

Example 4. Let $(A, \preccurlyeq)$ be a partially ordered set and let $B \subseteq A$. Let $\preccurlyeq$ also denote the restriction of $\preccurlyeq$ to B. Then $\preccurlyeq$ is a partial order on B, and $(B, \preccurlyeq)$ is a partially ordered set.

Definition 3. Let $(A, \preccurlyeq)$ and $(B, \preccurlyeq)$ be partially ordered sets. An *order morphism* from A to B is a function $f : A \to B$ satisfying

$$a_1 \preccurlyeq a_2 \Rightarrow f(a_1) \preccurlyeq f(a_2)$$

for all $a_1, a_2 \in A$. An *order isomorphism* from A to B is a bijective order morphism $f : A \to B$ such that f^{-1} is also an order morphism. If there exists an order isomorphism from A to B, we say that A and B are *isomorphic* as partially ordered sets, and write $A \cong B$.

Example 5. Let $f : \mathbb{N} \to \mathbb{N}$ be given by $f(n) = 2n + 3$. Then f is an order morphism from $(\mathbb{N}, \leq)$ to $(\mathbb{N}, \leq)$. We see that f is injective but is not surjective, so it is not an isomorphism.

Example 6. Let X and Y be sets and let $f : X \to Y$ be a function. This induces a function $F : \mathcal{P}(X) \to \mathcal{P}(X)$ which is an order morphism.

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Example 7. The sets Div(24) and Div(30) each have eight elements, so there is a bijective function between them. However, they are not isomorphic. On the other hand, Div(105) also has eight elements, and Div(30) $\cong$ Div(105).

Example 8. Let $n \in \mathbb{N}$. Then n factors into the product of a finite number of primes, and it is possible to write

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where p_i is prime for all i, and $1 \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_k$. Define the *prime shape* of n to be the ordered tuple $\sigma(n) = (\alpha_1, \dots, \alpha_k)$. Then $\mathrm{Div}(m) \cong \mathrm{Div}(n)$ if and only if $\sigma(m) = \sigma(n)$.

Definition 4. A *total order* on a set A is a partial order $\preccurlyeq$ with the additional property

(TO) $a \preccurlyeq b$ or $b \preccurlyeq a$.

A *totally ordered set* $(A, \preccurlyeq)$ is a set A together with a total order $\preccurlyeq$ on A.

Example 9. The real numbers $\mathbb{R}$, together with its standard ordering, is a totally ordered set.

Example 10. Let p be a prime number. Then $(\mathrm{Div}(p), |)$ is a totally ordered set.

Example 11. Two finite totally ordered sets are isomorphic if an only if they have the same number of elements.

Example 12. Any subset of a totally ordered set is totally ordered

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